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LETTER TO THE EDITOR

A model for which the derivatives of H alternate in sign

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Abstract. A recent proof by Simons that the H function is convex for a system of mono-energetic particles scattering off fixed centres when the scattering matrix is isotropic is extended to show that the derivatives of H alternate in sign.

The monotonic approach to equilibrium described by Boltzmann's H theorem (Harris 1971a) has been shown to be a characteristic of a wide class of kinetic equations. Often this is the only exact known result for the solution of these equations. The conjecture that H is convex (McKean 1966) has only been demonstrated for a very limited subset of this class and the more general statement that $(-1)^n \mathrm{d}^n H/\mathrm{d}t^n \equiv (-1)^n H^{(n)} \geqslant 0$ has been shown for only a few special cases. A fairly complete list of references has been given by Simons (1976) who has added two more cases to the former subset. In this letter we will focus on the second of these, the scattering of particles having the same energy by fixed centres. Simons (1976) was only able to prove the convexity property for this system when the scattering is isotropic, i.e. when the scattering operator $T_{ab} = T$. Our purpose here is to show the full alternating property holds for this model.

The number of particles in state a is f_a, the H function is $H = \Sigma_a f_a \ln f_a$, and the evolution of f_a is given by

$$f_a^{(1)} = N^{-1} \sum_b (f_b - f_a) \tag{1}$$

for isotropic scatter, where N is the number of states and the time has been suitably scaled. Since $\Sigma_b f_b$ is constant (the total number of particles), differentiating equation (1) with respect to time leads to

$$f_a^{(2)} = -N^{-1} \sum_b f_a^{(1)} = -f_a^{(1)} \tag{2}$$

using the definition of N. The condition $f_a^{(2)} = -f_a^{(1)}$ is sufficient, with the known result $H^{(1)} \leqslant 0$, to show that $(-1)^n H^{(n)} \geqslant 0$. The proof, which is inductive, will not be reproduced here but may be found in the literature in a general form (Harris 1971b). Since this last reference may not be widely available we note that a general proof may also be inferred from our result for the discrete velocity gas (Harris 1967, Gatignol 1975). Whether the alternating property, or even the convexity property, can be shown to be valid in a more general context, e.g. for the spatially uniform Boltzmann equation, remains debatable, but in our opinion of continued interest.

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